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A new finite-size scaling approach to random walks

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Abstract. We present a new finite-size scaling method for the random walks (RW) superseding a previously widely used renormalization group approach, which is shown here to be inconsistent. The method is valid in any dimension and is based on the exact solution for the two-point correlation function and on finite-size scaling. As an example, the phase diagram is derived for the random walk in two dimensions with a surface–bulk interaction where the system has either a surface or a defect line. We also discuss an initial calculation of the corresponding phase diagram for the case of a critically diluted lattice.

1. Introduction

The renormalization group (RG) has been a cornerstone in the analytical evaluation of the critical exponents of various statistical models in the past two decades [1–3]. Its application to self-avoiding walks (SAW) was a natural consequence of the formulation of SAW in the $n \to 0$ limit of the n-vector model whose two-point correlation function yields, in this limit, the generating function of the random walk with the excluded volume effect [4].

Ordinary random walks without the excluded volume effect are generally straightforward to characterize quantitatively; however, no exact solution exists when they are confined to disordered space, even in the absence of self-avoidance. This prompted searching for the extension of the RG approach to the random walks with no self-avoidance. Some RG approaches were constructed for the random walk in the presence of a quenched distribution of waiting times ([5] and references therein), which is a framework somewhat different from the one treated here. The best kinds of RG for including disorder appeared to be variants of real-space cell renormalization. For example, Family and Gould (FG) [6] studied a cell RG in the absence of disorder, while Sahimi and Jerauld (SJ) [7] and Gould and Kohin (GK) [8] worked on the disordered case, where the disorder was mimicked by a site (or bond) percolation cluster. The results in the latter cases were found to be in very good agreement with those of numerical simulations [9] of random walks on a percolating cluster on the square lattice. However, unlike GK who used the *kinetic* rule for the random walk (more precisely they solved the model which is nicknamed *myopic ant*) [10], SJ used the *static* recipe which is now recognized to correspond to the so-called *ideal chain* model [11]. Recent analysis [11–15] with various different approaches have shown that the ideal chain in a non-homogeneous environment does not belong to the same universality class as the kinetic walks; rather it is equivalent to the random walk in a trapping environment [11, 14].

By calculating exactly the two-point correlation functions for a random walk in a square cell of various sizes using the so-called *corner rule* [4], we will show that the general

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procedure introduced by FG needs a fundamental revision. This is also supported by exact analytical calculations on the one-dimensional analogue where an arbitrary cell size can be used. We will then proceed to show how one can obtain a consistent procedure based on the finite-size scaling hypothesis [16], which can then be applied to a variety of problems including the cases where the available space is disordered.

The outline of this paper is as follows. In section 2 we first recall the basis of the real-space renormalization group approach and discuss its fundamental problems. In this section, we also give the exact analysis of the one-dimensional versions of this problem, which helps to expose the inconsistencies of the renormalization approaches. In section 3, we introduce our new method which is based entirely on the widely accepted finite-size scaling approach and demonstrate it by calculating the phase diagram for a problem with different fugacities for the *surface* and *bulk*. We also describe the results of our initial calculations for the case where the space available to the random walk is critically diluted. A brief summary is given in section 4.

2. Difficulties with real-space renormalization procedures

It is well known that the unconstrained random walk on a lattice can be solved by using a generating function technique pioneered by Montroll [18–20]. Let $P_{x_0,x}(N)$ be the probability for the walker to be at the position $x \in \mathbb{Z}^d$ at the (discrete) time N, given that it started at the site x_0 at the initial time 0. The master equation to be solved is then

$$P_{x_0,x}(N+1) = \frac{1}{z}\sum_{y(x)} P_{x_0,y}(N) \tag{2.1}$$

where the notation $y(x)$ means that the sum is restricted to the nearest neighbours of x. The number $C_{x_0,x}(N) = z^N P_{x_0,x}(N)$ of N-step walks with end-points x_0 and x then satisfies the analogous equation

$$C_{x_0,x}(N+1) = \sum_{y(x)} C_{x_0,y}(N)\,. \tag{2.2}$$

In order to solve the master equation, it proves convenient to introduce the generating function $G_{x_0,x}(k)$,

$$G_{x_0,x}(k) = \sum_{N=0}^{\infty} k^N C_{x_0,x}(N) = \sum_{w:x_0\to x} k^{|w|} \tag{2.3}$$

where w is a walk having x_0 and x as the end-points, and $|w|$ is the number of steps associated with it.

By multiplying (2.2) by k^{N+1} and summing over all N, one gets, taking into account the initial condition $C_{x_0,x}(0) = \delta_{x_0,x}$,

$$G_{x_0,x}(k) = k\sum_{y(x)} G_{x_0,y}(k) + \delta_{x_0,x}\,. \tag{2.4}$$

It is easy to see that $G_{x_0,x}(k)$ is also the two-point correlation function of a scalar free-field theory and that (2.3) can be recovered from a von Neumann expansion (see e.g. [14, 20]).

Generally the procedure for a RG includes two basic steps: first one coarse grains microscopic details in real space or integrates over the *fast* modes in momentum space. This is followed by a rescaling of the space/momentum and of the model variables while conserving the partition function and recasting the Hamiltonian in the same functional form as before.

In the case of the random walk, the requirement of the conservation of the partition function amounts to a mapping between the rescaled and original fugacities which can be formally expressed as

$$P(\{k'\}, w') = \sum_{w|w'} P(\{k\}, w) \tag{2.5}$$

where k' and w' are the fugacity and walk on the rescaled lattice, respectively, P stands for the partition function, and the sum is constrained to all w consistent with w'. In the case of self-avoiding walks, this procedure leads to a well known polynomial recursion relation between k' and k whose linearization around the fixed point leads to the value of the correlation length exponent ν. On the other hand, once the self-avoidance is turned off, the polynomial recursion becomes an infinite series since there are an infinite number of walks even in the smallest possible cell. One is thus faced with the problem of finding either a way of summing over an infinite number of walks or a truncation procedure. Some time ago Family and Gould [6] devised a recipe along the latter line. Their idea was that if $L = ba$ (a being the lattice constant) is the size of the system, then walks with a number of steps N larger than N_{MAX} given by

$$N_{\mathrm{MAX}} \sim [\langle R_N^2 \rangle]_{\mathrm{MAX}} \sim L^2 \tag{2.6}$$

will give a negligible contribution to the sum in (2.5). As an example, if we renormalize from $L = 2a$ to $L' = a$, then we have

$$k' = k^2 + 2k^3 + 5k^4 + 14k^5 . \tag{2.7}$$

On the other hand the total number of N-step walks having x_0, x as the end-points can be calculated easily from (2.4) which gives the quantity (2.3) exactly. Therefore both sides of (2.5) can be calculated exactly without any truncation procedure. In figure 1 we compare the generating function calculated exactly with that calculated using the FG truncation procedure, which can clearly be recovered on numerical Taylor expansion of the exact result up to the desired order. It is apparent that, although in general the FG truncation procedure seems to reproduce rather well the trend of the fixed point k^*, it fails to reproduce the singularity present in the generating functions. This singularity moves closer and closer to the fixed point as the cell size increases. The physical origin of this singularity stems from the fact that, unlike other systems where criticality is reached only in the infinite volume limit, the random walk has a criticality in any finite cell, by taking the limit $N \to \infty$. This has a consequence, as seen in table 1, that the value of the exponent ν already overshoots the exact value $\frac{1}{2}$ at a very small cell size. This would also be the case with the FG truncation recipe if the size of the cell were pushed to a sufficiently large value (although the behaviour of such an approximation scheme for very large cell size is not known and may be complicated, see [17]).

It is also possible to do the exact calculation of the corner rule renormalization for *all* cell sizes in one dimension. Let us first consider a one-dimensional lattice where $x = 0, 1, 2, \ldots, L$ and the sites $x = -1$, $L + 1$ have an infinitely repulsive barrier. (The lattice constant a is set equal to 1 for simplicity.) The analogue of (2.4) for the correlation function for $x \geqslant 1$ is

$$G_{0,x}(k) = k[G_{0,x-1}(k) + G_{0,x+1}(k)] \tag{2.8}$$

along with the boundary conditions

$$G_{0,0}(k) = kG_{0,1}(k) + 1 \tag{2.9}$$

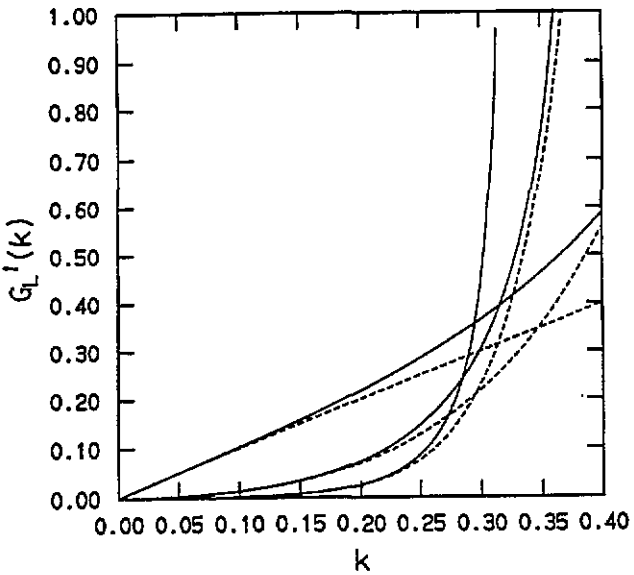

Figure 1. Comparison between the exact and approximate $G_L^{\uparrow}(k) \equiv \chi_2(k, 1/L)$ in the cases $L = 1, 2, 3$. The three full lines correspond to the exact evaluations, while the two broken lines are the approximate results as discussed in the text. The intersections of the full lines correspond to the exact fixed points $k^* = 0.3157, 0.2950$, while the intersections of the broken lines correspond to the Family–Gould fixed points $k_{FG}^* = 0.3470, 0.3108$ for a scaling from $L = 2, 3$ to $L' = 1, 2$, respectively.

Table 1. Behaviour of ν as functions of the cell size $b = L/a$, where a is the lattice constant. The first two columns refer to the present work, while the second two refer to the results using the approximate recipe of Family and Gould [6].

Scaling lengths b/b'	k^*	ν	k_{FG}^*	ν_{FG}
$\frac{2}{1}$	0.3156	0.5438	0.3470	0.5853
$\frac{3}{1}$	0.2950	0.5132	0.3108	0.5571
$\frac{3}{2}$	0.2770	0.4441	0.2920	0.5129
$\frac{4}{1}$	0.2825	0.4937	0.2926	0.5412
$\frac{4}{3}$	0.2711	0.4485	0.2743	0.4868
$\frac{5}{1}$	0.2745	0.4792	0.2838	0.5398
$\frac{5}{4}$	0.2640	0.4351	0.2693	0.5148

and

$$G_{0,L}(k) = kG_{0,L-1}(k) \tag{2.10}$$

where we have assumed that all walks start from $x = 0$.

We can put this equation in a transfer matrix form,

$$\Psi_x(k) = \mathbf{T}\Psi_{x+1}(k) \qquad x = 1, 2, \ldots, L-1 \tag{2.11}$$

where we defined

$$\mathbf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 1/k \end{pmatrix} \qquad \Psi_x = \begin{pmatrix} G_{0,x}(k) \\ G_{0,x-1}(k) \end{pmatrix} \tag{2.12}$$

The eigenvalues of the matrix **T** are clearly

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1-4k^2}}{2k} \tag{2.13}$$

which are real if $0 < k \leqslant \frac{1}{2}$ and form a complex conjugate pair if $\frac{1}{2} < k < 1$ (we consider $k < 1$ in order to make the generating functions sensible). From these and the boundary conditions, we find

$$G_{0,L}(k) = \frac{\lambda_+ - \lambda_-}{\lambda_+^{L+1} - \lambda_-^{L+1} - k(\lambda_+^L - \lambda_-^L)}. \tag{2.14}$$

Note that, although this solution is valid for arbitrarily large L, the boundary condition (2.10) makes the system strictly *finite*.

This expression for $G_{0,L}(k)$ gives the recursion relation

$$\chi_{\hat{1}}(k', 1/L') = \chi_{\hat{1}}(k, 1/L). \tag{2.15}$$

Using (2.14) and (2.15) for renormalization from a cell of size $L+1$ to that of size L, we get an implicit solution for the fixed point $k^*(L+1, L)$,

$$k_L^* \equiv k^*(L+1, L) = \frac{\lambda_+^{*L+2} - \lambda_-^{*L+2} - (\lambda_+^{*L+1} - \lambda_-^{*L+1})}{\lambda_+^{*L+1} - \lambda_-^{*L+1} - (\lambda_+^{*L} - \lambda_-^{*L})} \tag{2.16}$$

where $\lambda_{\pm}^*$ is $\lambda_{\pm}$ evaluated at $k = k_L^*$.

After some algebra, we can obtain an explicit solution for the fixed point for asymptotically large L:

$$k_L^* = \frac{1}{2} + \left(\frac{\pi}{4}\right)^2 \frac{1}{L^2} + O\left(\frac{1}{L^3}\right). \tag{2.17}$$

The exact critical value k_c is obtained from k_L^* in the $L \to \infty$ limit, as expected. However, linearizing the recursion relation (2.15) around the fixed point, we get the eigenvalue

$$\Lambda_L = \left.\frac{\partial k'}{\partial k}\right|_{k=k_L^*} = \left.\frac{\partial G_{0,L+1}(k)/\partial k}{\partial G_{0,L}(k')/\partial k'}\right|_{k=k'=k_L^*} \tag{2.18}$$

$$= 1 + \left(2 + \frac{\pi^2}{4}\right)\frac{1}{L} + O\left(\frac{1}{L^2}\right) \tag{2.19}$$

and thus

$$\nu = \lim_{L\to\infty} \frac{\ln[(L+1)/L]}{\ln \Lambda_L} = \frac{1}{2 + \pi^2/4} = 0.2238\ldots. \tag{2.20}$$

Thus the exponent ν does *not* have the correct limiting value (which is $\frac{1}{2}$), but rather badly overshoots $\frac{1}{2}$ in the same way as in the numerical evaluation in two dimensions.

One might argue that the incorrect $L \to \infty$ trend of the eigenvalue Λ_L may be due to the fact that the above cell RG procedure uses a strictly finite *system* size L (however large it may be) due to the infinitely repulsive boundary condition at both ends of the one-dimensional cell. However, we can see that this is not the case by renormalizing a one-dimensional generating function with a semi-infinite boundary condition, i.e. with an infinitely repulsive boundary only on one side. The exact calculation in this case gives the fixed point at $k^* = \frac{1}{2}$ exactly independent of L or L', which is consistent with $k_c = \frac{1}{2}$, but the eigenvalue $\Lambda(L, L')$ is

$$\Lambda(L, L') = \frac{L+1}{L'+1}. \tag{2.21}$$

This is clearly wrong, since it would yield $\nu = 1 + 0(1/L)$. As it turns out, $G_{0,L}(k)$ is singular at $k = \frac{1}{2}$, and what we are attempting to do is to linearize around a singular fixed point.

Even though the real-space renormalization approach described above results in a nonsensical exponent value, the result for the two-point correlation function $G_{0,L}(k)$ itself is correct. Indeed it is easy to check that it reproduces the exact results for the critical exponents ν and γ_1 (a surface exponent [16]) if they are calculated directly from the correlation function rather than from the recursion relation based on it. More specifically one finds in the grand-canonical ensemble,

$$\chi_1(k) = \sum_{L=0}^{\infty} G_{0,L}(k) \overset{k\to\frac{1}{2}^-}{\sim} (1-2k)^{-1/2} \tag{2.22}$$

which is the exact result for the surface exponent $\gamma_1 = \frac{1}{2}$ (see next section) and

$$\xi^2(k) = \frac{\sum_{L=0}^{\infty} L^2 G_{0,L}(k)}{\sum_{L=0}^{\infty} G_{0,L}(k)} \overset{k\to\frac{1}{2}^-}{\sim} (1-2k)^{-1} \tag{2.23}$$

which again gives the exact result $\nu = \frac{1}{2}$.

3. Finite-size scaling and surface critical behaviour

We learned from the previous calculations that, in the usual form, the general cell renormalization procedure for the random walk used in [6–8] cannot be consistent. That is, it is not assured that, as the size of the cell increases, the results for the critical exponents improve and become exact in the limit of an infinite cell. This is built in to the general method itself and not specific to any particular truncation procedure used in getting an explicit recursion relation. If an improvement is attempted by using exact correlation functions, an inconsistency is found stemming from the basic recursion relation.

The clue of how to remedy the situation comes from the result (2.17) where we calculated *how* the fixed point was becoming exact in the $L \to \infty$ limit. This is indeed compatible with the finite-size scaling hypothesis (see e.g. [16] and references therein),

$$k_c(L) = k_c(\infty) + A\left(\frac{1}{L}\right)^{1/\nu} \tag{3.1}$$

where $k_c(\infty) = \frac{1}{2}$ is the exact critical fugacity in the infinite-lattice limit and $\nu = \frac{1}{2}$ in this case. The idea, therefore, is that one can estimate $k_c(L)$ by looking at the divergences of the (bulk) susceptibility defined as

$$\begin{aligned}\chi_B(k) &= \frac{1}{|\Lambda|} \sum_{x_0\in\Lambda} \sum_{N=0}^{\infty} C(x_0, N; \Lambda) k^N \\ &= \frac{1}{|\Lambda|} \sum_{x_0\in\Lambda} \sum_{x\in\Lambda} G_{x_0,x}(k)\end{aligned} \tag{3.2}$$

where $C(x_0, N; \Lambda)$ is the number of N-step walks starting from a point x_0 and entirely contained in the volume Λ and $G_{x_0,x}(k)$ is its generating function. The subscript B refers to *bulk* in the sense that the end-points x_0 and x can be anywhere in volume Λ. Then, either by fixing the exact value of $\nu = \frac{1}{2}$ one can calculate $k_c(\infty)$ or by fixing the exact value of $k_c(\infty)$ one can calculate the value of the exponent ν.

The results for the square lattice are shown in figures 2 and 3, and they are consistent with the expected values. Indeed a best fit for both cases gives $k_c(\infty) = 0.25 \pm 0.01$ and

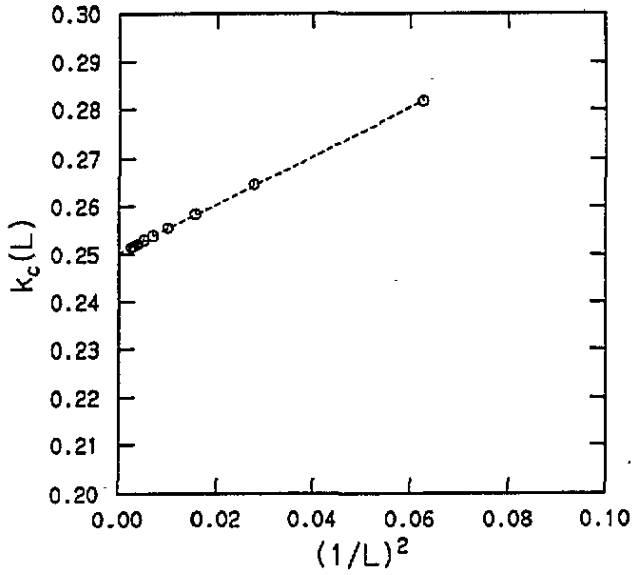

Figure 2. Finite-size scaling result for the critical value $k_c(\infty)$, which corresponds to the true critical point for the square lattice. A best fit over all points gives $k_c(\infty) = 0.25 \pm 0.01$, while the exact value is $k_c = \frac{1}{4}$.

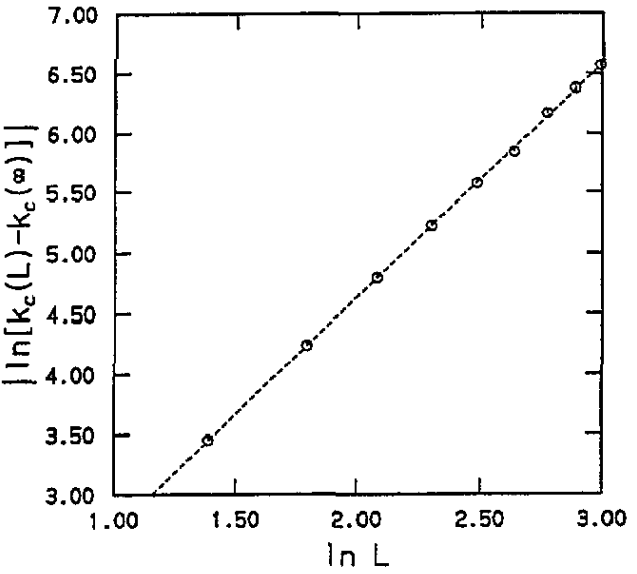

Figure 3. Finite-size scaling result for the exponent ν. A best fit over all points gives $1/\nu = 1.94 \pm 0.02$, while the exact value is $1/\nu = 2$.

$1/\nu = 1.94 \pm 0.02$ and in the case of ν improves if we include more and more terms corresponding to larger cell sizes.

The presence of the surface also changes the entropic critical exponents, as is well known [16]. Indeed, if the system is sufficiently large to make the distinction between

surface and bulk sensible, one can decompose the total free energy F_L as

$$F_L\left(\Delta k, h, h_1, \frac{1}{L}\right) \overset{L\gg 1}{\approx} L^d f_B\left(\Delta k, h, \frac{1}{L}\right) + L^{d-1} f_S\left(\Delta k, h, h_1, \frac{1}{L}\right) \quad (3.3)$$

where h and h_1 are the external fields associated with the bulk and the surface, respectively, and $\Delta k = k - k_c$. By differentiating twice with respect to the proper external field and by using the finite-size scaling ansatz, one gets the well known general results

$$\hat{\chi}_B(L) \overset{L\gg 1}{\sim} L^{\gamma/\nu} \qquad \hat{\chi}_1(L) \overset{L\gg 1}{\sim} L^{\gamma_1\nu} \qquad \hat{\chi}_{1,1}(L) \overset{L\gg 1}{\sim} L^{\gamma_{1,1}\nu} \quad (3.4)$$

at the critical values $\Delta k = 0$, $h = h' = 0$. Here we have defined the *local* susceptibilities:

$$\chi_1(k) = \frac{1}{|\partial\Lambda|} \sum_{x_0\in\partial\Lambda} \sum_{x\in\Lambda} G_{x_0,x}(k) \qquad \chi_{1,1}(k) = \frac{1}{|\partial\Lambda|} \sum_{x_0\in\partial\Lambda} \sum_{x\in\partial\Lambda} G_{x_0,x}(k) \quad (3.5)$$

where we mean by $\partial\Lambda$ the boundaries of the volume Λ.

Thus, our new method forgoes the usual corner rule *renormalization per se*, and instead, calculates various quantities associated with a finite cell and interprets them in terms of *surface* finite-size scaling. From figure 4 one can see that this method reproduces very accurately the expected values predicted, for example, by the mean-field theory [16, 21], namely $\gamma/\nu = 2$, $\gamma_1/\nu = 1$ and $\gamma_{1,1}/\nu = -1$.

As an example of the method just described, we present the finite-size scaling solution of the problem of the interplay between the bulk Λ and the surface $\partial\Lambda$. In this problem, each step of the random walk acquires fugacity k if it is anywhere except on the boundary line, and fugacity k_1 if it lies on the boundary line, and we perform the exact calculation of the finite cell susceptibilities. The random walk with a reflecting or absorbing boundary is of course a classic exactly solvable problem [22]; there also exist exact treatments of the problem of the effects of the attractive boundary where the degree of attraction is varied

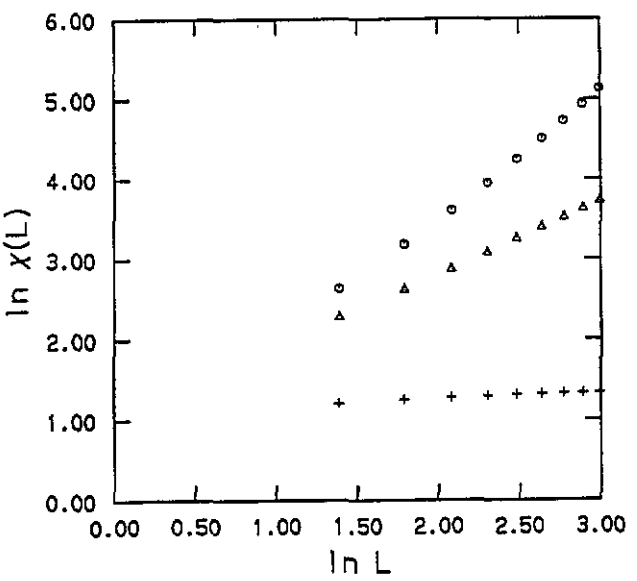

Figure 4. Evaluation of the bulk and surface susceptibilities in the finite-size scaling approach. The estimates for χ_B (○), χ_1 (△) and $\chi_{1,1}$ (+) are obtained from the slopes of the lines shown in the log–log plots. The exact values according to the mean-field calculation are $\gamma/\nu = 2$, $\gamma_1/\nu = 1$ and $\gamma_{1,1}/\nu = -1$.

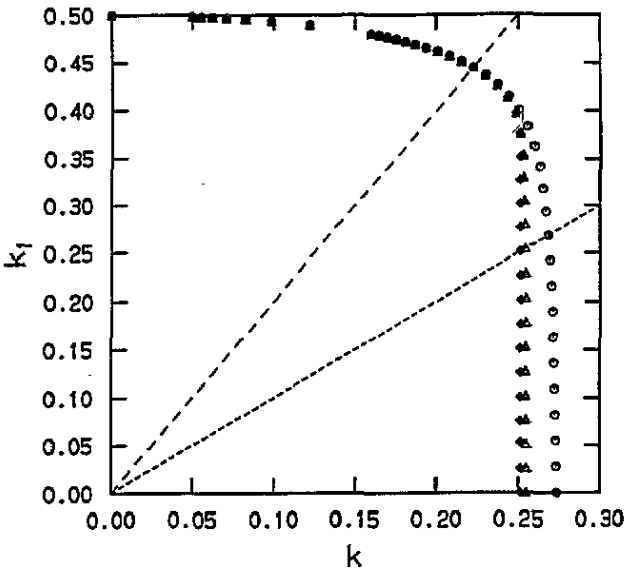

Figure 5. Computed phase diagram for the surface–bulk problem. The points shown have been calculated for different system sizes, $L = 4$ (○), $L = 10$ (△), and $L = 20$ (◇). The dotted and broken lines correspond to slopes $k_1/k = 1, 2$, respectively. The special point is estimated to be at $k = 0.25 \pm 0.01$, $k_1 = 0.35 \pm 0.01$.

(e.g. [23]). The latter is, however, not a trivial problem and the exact treatments tend to be rather involved. The present approach is much simpler and yet yields accurate results and serves to illustrate the power of the method.

Physically the possibility of changing the strength of the surface fugacity with respect to the bulk fugacity allows the surface to *make up* for the missing bonds. Clearly one expects that if k_1 is sufficiently strong almost all walks lie on the surface, and then the critical point and the universality class should both change: when all the *interactions* in the bulk are zero, the walks are not allowed to stay in the bulk and we have the *adsorbed* phase. Since $k_1 = \frac{1}{2}$ and $k = \frac{1}{4}$ are the exact critical values corresponding to an infinite *surface* (a line in this case) and an infinite bulk, one then expects a qualitative phase diagram, as shown in figure 5.

For this calculation we eliminated the *corner* and imposed a periodic boundary condition into the vertical direction $\hat{2}$, while the horizontal direction $\hat{1}$ is of size L and has free edges. The result is shown in figure 6. It appears that there is a tricritical point (called a *special point*) which is the intersection of three different lines (corresponding to three different second-order phase transitions). Below the special point there is the *ordinary* transition, where the bulk and the surface undergo a transition at the same critical point. Above the special point there is a line of *surface* transitions, which take place if k_1 is bigger then the special point ordinate, where the surface goes into an *ordered* state (where the susceptibility is singular) while the bulk is still *disordered*, as well as another line called the *extraordinary* line where the bulk also becomes *ordered*.

Our estimate of the special point SP is at $k = 0.25 \pm 0.01$, and $k_1 = 0.35 \pm 0.01$, corresponding to a ratio $k_1/k = 1.40 \pm 0.06$. This is in reasonably good agreement with a simple mean-field argument which would predict the ratio of $\frac{4}{3}$. The errors were estimated graphically.

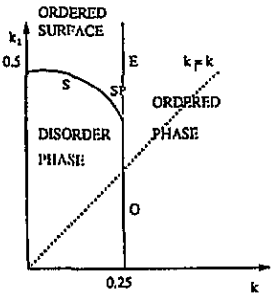

Figure 6. A sketch of the expected phase diagram. The lines shown correspond to the ordinary transition (O), the surface transition (S) and the extraordinary transition (E), and the point labelled SP is the tricritical point called the special transition.

The qualitative features contained in this phase diagram also appear in the case of the self-avoiding walks [24] and in percolation [25]. Quite similar features are found in the case of a defect (which is a $(d-1)$-dimensional surface inserted into a d-dimensional bulk) as well. In this case, the special point SP is found when $k_1/k = 1$, again consistent with mean-field arguments.

A second and perhaps more interesting example is the case where the random walk is constrained to a randomly diluted square lattice. We first applied our variant of the real-space cell renormalization to the diluted case. In this case, we enumerate all configurations of the cell with respective probabilities of occurrence at the critical dilution (i.e. when each bond is present with a probability of about $p = 0.5$), and on each configuration. For the *ideal chain* case we found, from a numerical Taylor expansion up to the proper order, $\nu = 0.606$ and $\nu = 0.516$ for $L = 2, 3$ and $L' = 1, 2$, respectively, in very good agreements with the values presented in [7]. The exact calculations, however, give values which get worse and worse as the cell size increases, as expected. For the *myopic ant* we found similar consistency with the results of [8], given that they treated *site* percolation and used a different truncation procedure.

We then extended the analysis of the surface–phase phase diagram to the disordered $L = 2$ cell only, and leave a more comprehensive study of this problem to a subsequent work. We calculated the surface and bulk susceptibilities exactly in the same way as described above. The locations of the singularities in the ensemble-averaged susceptibilities then determine the $L = 2$ approximations to the phase boundary.

The results for the *myopic ant* rule are given in figure 7. One notes immediately an unusual feature where two separate boundaries are traced from the bulk and surface susceptibilities, although they appear to merge for small values of k. In particular, it appears at this level that the bulk susceptibility may have a singularity before the surface susceptibility becomes singular. Of course this does not mean that the true *bulk* ordering occurs before surface ordering because there is a substantial *surface* contribution even in the *bulk* susceptibility as we have calculated it. This may simply be a spurious effect of the small cell size. Nevertheless, this might also suggest a qualitatively different situation, as such discrepancies did not arise in the non-diluted case even for very small cell sizes. It may be noted that the susceptibility singularity for an ideal chain is not expected to be a simple power law in strongly correlated disorder but rather an *essential* singularity [12–15]. More work is clearly needed to elucidate this situation, which is beyond scope of this paper.

4. Conclusions

In this paper we have given an analysis, both in the undiluted and diluted case, of the difficulties associated with the usual cell renormalization approach to the random-walk

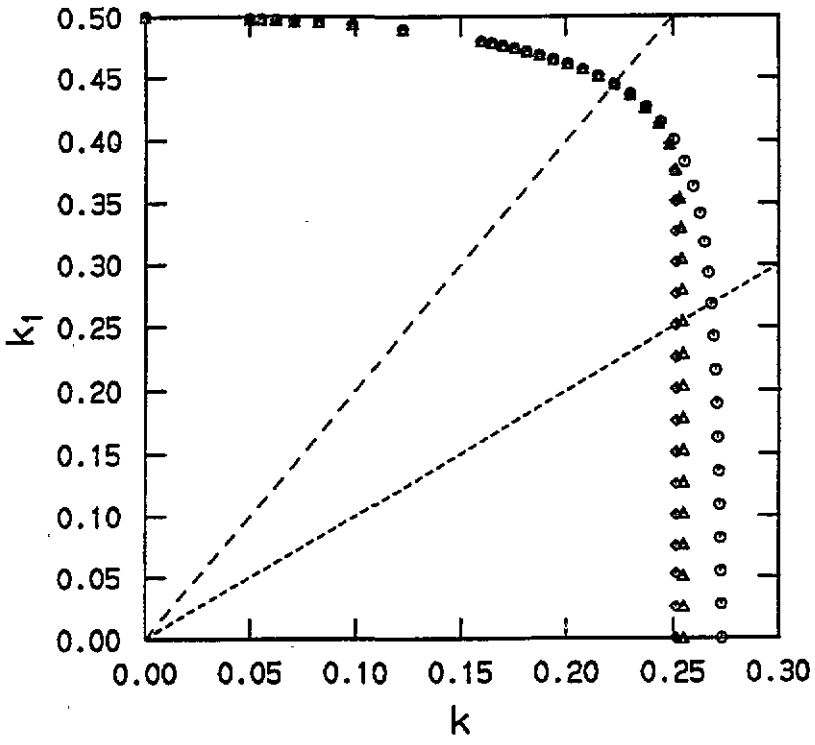

Figure 7. Computed phase diagram for the surface–bulk problem for the random walk on a 2×2 cell for the diluted case for bond ($p = 0.5$). The rule chosen was the *myopic ant* described in the text with periodic boundary conditions in the $\hat{2}$ direction (parallel to the surface), and open boundary conditions in the $\hat{1}$ direction, where only those walks which span were averaged. The symbols ($\triangle$) and ($\square$) correspond to the bulk and surface singularities as explained in the text.

problem (with or without the Family–Gould truncation procedure [6]), and presented an alternative method to calculate the critical properties which does not suffer from similar difficulties. This new approach is shown to give results which continually improve as the size of the cell increases, unlike in the previous approaches which are shown here not to have this essential feature. Our approach is based only on the finite-size scaling hypothesis. We have demonstrated this approach by first calculating the full phase diagram of the effect of the surface fugacity having a different value from the bulk fugacity, and calculating the exponents γ_1, $\gamma_{1,1}$ and ν. Secondly, we have also presented an initial calculation of the corresponding phase diagram for the random walk constrained to a randomly diluted substrate. Although this problem is potentially important as it has the essential features of transport through disordered media (an important materials problem), reliable investigations are lacking when the disorder is strongly correlated.

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